Hydrodynamics of Gaseous System in Massive Brans-Dicke Gravity

M. Sharif [1] *and Rubab Manzoor [1,2] †

[1] Department of Mathematics, University of the Punjab,
Quaid-e-Azam Campus, Lahore-54590, Pakistan.
[2] Department of Mathematics,
University of Management and Technology,
Johar Town Campus, Lahore-54782, Pakistan.

Abstract

This paper explores hydrodynamics and hydrostatic of a star in post-Newtonian approximation of massive Bran-Dicke gravity. We study approximated solution of the field equations upto $O(c^{-4})$ and generalize Euler equation of motion. We then formulate equations governing hydrodynamics, stability and instability of the system. Finally, we discuss spherically symmetric stars for a specific barptropic case like dust, cosmic string and domain wall in this scenario.

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1   Introduction

The mysteries of the universe always remain interesting issue for the physicists. The combination of two theories, i.e., theory of fluid dynamics and

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*msharif.math@pu.edu.pk
†rubab.manzoor@umt.edu.pk

theory of gravity play a major role in the study of the universe. It describes universe phenomena in three contexts: (i) dynamics of celestial objects in weak-field regimes (like dynamics of solar system), (ii) dynamics of strong-field regimes such as Supernovae, formation of black hole, superluminal jets, binaries pulsars, neutron stars, X-rays and Gamma-rays burst etc and (iii) cosmological issues (origin, different eras and eventual final fate of the universe) [1].

In this context, the concept of weak-field approximation (Newtonian and post-Newtonian (pN) approximations) of relativistic hydrodynamics has taken considerable importance in many regards. It provides a platform where we can evaluate ranges of deviation and level of consistency between relativistic theory of gravity and Newton's gravity. The analysis of phenomena in strong-field regimes are highly complicated and non-linear, so in order to obtain results of physical interest, weak-field approximation schemes are used as an effective tool [2]. Chandrasekhar [3] combined theory of hydrodynamics and general relativity (GR) at pN limits to remove difficulties occurring in the analysis of large scale structures. He derived solutions of the field equations in pN correction (order of c^{-4}) of GR which contain potential and superpotential functions to represent masses of gaseous or stellar structure. He then used these solutions to derive a set of generalized Newtonian hydrodynamics equations and discussed complicated issues of gaseous structures (like radial as well as non-radial oscillation of stars, rotating homogenous masses and stability of gaseous masses under non-radial as well as radial oscillation) in a approximated region in which strong field interactions (gravitational radiation) play no role. In this way, he represented a mechanism (in weak-field approximation) which can analyze different gaseous or stellar systems in GR [4]. After his work, many researchers [5] explored different astrophysical systems in weak-field approximation of GR and obtained many interesting results.

The modified theories of gravity are considered the most fascinated approaches to resolve mysteries of the present universe (the dark energy and dark matter). Brans-Dicke (BD) gravity [6] is an attractive example of modified gravity in which gravity is mediated by a massless scalar field ϕ and curvature. It contains a coupling constant ω_{BD} which serves as a tuneable parameter and can adjust results according to the requirement. This theory has provided convenient solutions of many cosmic issues but unable to explain "graceful exist" problem of old inflationary according to observational surveys. The inflationary model defined by BD gravity is valid for specific

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ranges of coupling parameter ($\omega_{BD} \leq 25$) [7] which is not consistent with observational limits [8]. Moreover, the BD gravity probes strong field test (cosmic issues) for negative and low values of ω_{BD} [11] but satisfies weak field test for high and positive values of ω_{BD} [9]. Thus, the weak field evaluations are not consistent with the results of strong field.

In order to resolve this issue, a scalar potential function $V(\phi)$ (massive function) is introduced in BD gravity which leads to massive Brans-Dicke (MBD) gravity [10]. This new theory not only solves cosmic issue (old inflation) but also provides a consistency between the results obtained for weak-field as well as cosmic scale [12]. There is a large body of literature which describes dynamics of the universe in modified gravity [13]. In this context, Nutku [14] explained fluid hydrodynamics in this gravity. He represented approximated solutions of BD equations in complete pN limits (order of c^{-4}) involving potential functions (representing masses of the celestial objects) and explored stability of spherically symmetric gaseous system. Olmo [15] evaluated complete pN limits of MBD field equations solutions but he converted only lowest-order (order of c^{-2}) approximation of solutions in terms of potential functions to discuss $f(R)$ gravity as a special case of scalar-tensor gravity.

In order to provide a suffice platform for the analysis of gaseous systems in accelerated expanding universe, we represent hydrodynamics of fluid in complete pN limits of MBD gravity. For this purpose, in section **1**, we obtain complete pN approximation (order of c^{-4}) of MBD field solutions in terms of potential and superpotential functions. In section **3** we formulate equations governing hydrodynamics of the fluid in pN regimes. Section **4**, evaluates conditions that govern stability and instability. Section **5** explores spherically symmetric barotropic stars in MBD gravity. Finally, section **6** summarizes the results.

2 Massive Brans-Dicke Gravity in Post-Newtonian Limits

The action of MBD gravity with ($\kappa^2 = \frac{8\pi G}{c^2}$) [12] is

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g}[\phi R - \frac{\omega_{BD}}{\phi}\nabla^\alpha\phi\nabla_\alpha\phi - V(\phi)] + L_m[g, \psi], \qquad (1)$$

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where $L_m[g, \psi']$ represents matter action which depends upon metric as well as matter field ψ'. Variation of the above action with respect to $g_{\alpha\beta}$ and ϕ provides respective MBD equations

$$G_{\alpha\beta} = \frac{\kappa^2}{\phi}T_{\alpha\beta} + [\phi_{,\alpha;\beta} - g_{\alpha\beta}\Box\phi] + \frac{\omega_{BD}}{\phi}[\phi_{,\alpha}\phi_{,\beta} - \frac{1}{2}g_{\alpha\beta}\phi_{,\mu}\phi^{,\mu}] - \frac{V(\phi)}{2}g_{\alpha\beta},$$
(2)

$$\Box\phi = \frac{\kappa^2 T}{3 + 2\omega_{BD}} + \frac{1}{3 + 2\omega_{BD}}[\phi\frac{dV(\phi)}{d\phi} - 2V(\phi)].$$
(3)

Here $T_{\alpha\beta}$ is the energy-momentum tensor of matter, $T = g^{\alpha\beta}T_{\alpha\beta}$ and $\Box$ is the d'Alembertian operator. Equations (2) and (3) describe the MBD field equations and evolution equation for the scalar field, respectively. We take matter contribution in the form of perfect fluid which is compatible with pN approximation

$$T_{\alpha\beta} = [\rho c^2(1 + \frac{\pi}{c^2}) + p]u_\alpha u_\beta - pg_{\alpha\beta},$$
(4)

where ρ, $\rho\pi$, p and u_μ represent matter density, thermodynamics internal energy, pressure and four velocity, respectively.

In order to calculate some approximated MBD solutions in pN regime, the following Taylor expansions are assumed [15]

$$g_{\alpha\beta} \approx \eta_{\alpha\beta} + h_{\alpha\beta}, \quad \phi \approx \phi_0 + \varphi^{(2)} + \varphi^{(4)},$$
$$V(\phi) \approx V_0 + \varphi V_0' + \varphi^2 V_0''/2 + ...$$

Here $\eta_{\alpha\beta}$ is the Minkoski metric (representing non-dynamical background), $h_{\alpha\beta}$ shows the perturbation tensor which describes deviation of $g_{\alpha\beta}$ from $\eta_{\alpha\beta}$, the term ϕ_0 indicates asymptotic cosmic function which slowly varies with respect to cosmic time t_0, $\varphi(t, x)$ represents local deviation of scalar field from ϕ_0 with superscripts ((2) and (4)) indicating order of approximation (c^{-2}) as well as (c^{-4}) and $V_0 = V(\phi_0)$ shows the value of potential function at time t_0. The pN limits of MBD theory are obtained through the gauge condition

$$h_{k,\alpha}^\alpha - \frac{1}{2}h_{\alpha,k}^\alpha - \frac{\partial_k\varphi}{c^2\phi_0} = 0,$$

and the lowest-order $(O(2))$ pN approximated solutions describing the potential of the compact object are given by

$$g_{00} \approx 1 - h_{00}^{(2)} = 1 - \frac{2U}{c^2} + \frac{\Lambda_{BD}r^2}{3c^2},$$
(5)

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$$g_{ij} \approx -[1 + h_{ij}^{(2)}]\delta_{ij} = [-1 - \frac{2\gamma_{BD}U}{c^2} - \frac{\Lambda_{BD}r^2}{3c^2}]\delta_{ij}, \tag{6}$$

$$\frac{\varphi}{\phi_0} \approx \frac{-2U}{c^2}\left[\frac{e^{-m_0 r}}{3 + 2\omega_{BD} + e^{-m_0 r}}\right], \tag{7}$$

where $i, j = 1, 2, 3$ and U is the gravitational potential determined by Poisson's equation

$$\nabla^2 U = -4\Pi\rho G_{eff} \tag{8}$$

with

$$G_{eff} = \frac{\kappa^2}{8\pi\phi_0}\left(1 + \frac{e^{-m_0 r}}{3 + 2\omega_{BD}}\right), \quad m_0 = \left(\frac{\phi_0 V_0'' - V_0'}{3 + 2\omega_{BD}}\right)^{1/2}.$$

The term $\Lambda_{BD} = \frac{V_0}{2\phi_0}$ shows the cosmological constant, γ_{BD} is the parameterized pN parameter given by

$$\gamma_{BD} = \frac{3 + 2\omega_{BD} - e^{-m_0 r}}{3 + 2\omega_{BD} + e^{-m_0 r}}.$$

To discuss characteristics of celestial fluid in complete pN limits $(O(4))$, we evaluate complete pN solutions of MBD equation in terms of potential and super-potential functions [3, 14]. For this purpose, we calculate values of $g_{00} \sim O(4)$, $g_{0i} \sim O(3)$ in terms of potential and super-potential functions. Consider the following metric coefficients

$$g_{00} = 1 + h_{00}, \quad g_{0i} = h_{0i}, \quad g_{ij} = -\delta_{ij} + h_{ij}, \tag{9}$$

where

$$h_{00} = h_{00}^{(2)} + O(4), \quad h_{0i} = O(3), \quad h_{ij} = h_{ij}^{(2)}.$$

The components of four-velocity and energy-momentum tensor can be found by using Eq.(9) given in appendix **A**. Equation (2) can also be written as

$$R_{\alpha\beta} = \frac{\kappa^2}{\phi}(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T) + \frac{\omega_{BD}}{\phi^2}[\phi_{,\alpha}\phi_{,\beta}] + \frac{1}{\phi}[\phi_{,\alpha;\beta}] - \frac{g_{\alpha\beta}}{2\phi}[\Box\phi + V(\phi)]. \tag{10}$$

The $(0, i)$ and $(0,0)$ components of above equation are given by [15]

$$\frac{-1}{2}\nabla^2 h_{0i}^{(3)} - \frac{1}{4}h_{00,0j}^{(2)} + 8\Pi G_{eff}\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}}\rho v_i = 0, \tag{11}$$

$$\frac{-1}{2}\nabla^2\left[h_{00}^{(4)} + \frac{(h_{00}^{(2)})^2}{2} + \frac{1}{2}(\frac{\varphi^{(2)}}{\phi_0})^2\right] = \frac{\kappa^2\rho}{2\phi_0}\left[c^2 + \pi + 2v^2 + h_{[ij]}^{(2)} - \frac{\varphi^{(2)}}{\phi_0}\right.$$

$$\left. + \frac{3p}{\rho}\right] - \frac{1}{2\phi_0}\left[V_0(1 + h_{[ij]}^{(2)} - \frac{\varphi^{(2)}}{\phi_0}) + \varphi^{(2)}V_0'\right]. \tag{12}$$

Here the effect of ϕ_0 is considered almost constant, hence the contributions due to $\dot{\phi}_0$ and $\ddot{\phi}_0$ are neglected.

In order to solve Eq.(11) for $h_{0i}^{(3)}$, we assume that the second and third term of the equation can be expressed in terms of the potential functions χ as well as U_i represented by the Poisson equations as

$$\nabla^2\chi = h_{00}^{(2)} = \frac{1}{c^2}(-2U + \frac{\Lambda_{BD}r^2}{3}), \tag{13}$$

$$\nabla^2\left(\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}}\right)U_i = -4\Pi G_{eff}\frac{3 + 2\omega_{BD}\rho v_i}{3 + 2\omega_{BD} + e^{-m_0 r}}, \tag{14}$$

where $U_i = -4\Pi G\rho v_i$ [3]. Equations (11), (13) and (14) give

$$\frac{-1}{2}\nabla^2 h_{0i}^{(3)} - \frac{1}{4}\nabla^2\frac{\partial^2\chi}{\partial t\partial x_i} - 2\nabla^2\left(\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}}\right)U_i = 0,$$

which provides

$$h_{0i}^{(3)} = \frac{1}{c^3}\left(4U_i\left(\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}}\right) - \frac{1}{2}\frac{\partial^2\chi}{\partial t\partial x_i}\right). \tag{15}$$

Now we verify that this solution (15) is consistent with gauge condition as

$$h_{0,\alpha}^{\alpha} - \frac{1}{2}h_{\alpha,0}^{\alpha} - \frac{\partial_0\varphi}{c^2\phi_0} = \frac{4}{c^3}\left[\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m0r}}\right]$$

$$\left[\frac{\partial U}{\partial t} + \frac{3 + 2\omega_{BD} + e^{-m_0 r}}{3 + 2\omega_{BD}}\frac{\partial}{\partial x_i}\left(\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}}\right)U_i\right]. \tag{16}$$

From Eqs.(8), (14) and Newtonian equation of continuity, we have

$$\nabla^2\left[\frac{\partial U}{\partial t} + \frac{3 + 2\omega_{BD} + e^{-m_0 r}}{3 + 2\omega_{BD}}\frac{\partial}{\partial x_i}\left(\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}}\right)U_i\right]$$

$$= \frac{-4\pi G_{eff}}{c^3}\left[\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i)\right] = 0,$$

which implies

$$\left[\frac{\partial U}{\partial t} + \frac{3 + 2\omega_{BD} + e^{-m_0 r}}{3 + 2\omega_{BD}} \frac{\partial}{\partial x_i} \left(\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}}\right) U_i\right] = 0. \qquad (17)$$

Equations (16) and (17) show the consistency of the solution with gauge condition.

Now we evaluate $h_{00}^{(4)}$ from Eq.(12). For this purpose, we consider that the left hand side of Eq.(12) can be defined in terms of potential function ψ and super-potential Φ given by the following Poisson equations [3, 14]

$$\nabla^2 \psi = -\frac{1}{4\phi_0}\left[V_0(1 + h_{[ij]}^{(2)} - \frac{\varphi^{(2)}}{\phi_0}) + \varphi^{(2)}V_0'\right], \qquad (18)$$

$$\nabla^2 \Phi' = -4\Pi G_{eff}\rho\sigma, \quad \Phi' = \Phi + 2\psi, \qquad (19)$$

where

$$\sigma = \frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}}\left[\pi + 2v^2 - h_{00}^{(2)} - \frac{\varphi^{(2)}}{\phi_0} + \frac{3p}{\rho}\right].$$

Equations (8), (12), (18) and (19) give

$$\nabla^2 \left[h_{00}^{(4)} + \frac{(h_{00}^{(2)})^2}{2} + \frac{1}{2}(\frac{\varphi^{(2)}}{\phi_0})^2 + \frac{2U}{c^4}(\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} - e^{-m_0 r}}) + 2\Phi + 2\psi\right] = 0$$

which implies

$$h_{00}^{(4)} = -\frac{(h_{00}^{(2)})^2}{2} - \frac{1}{2}(\frac{\varphi^{(2)}}{\phi_0})^2 - \frac{2U}{c^4}(\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}}) - 2\Phi - 2\psi. \qquad (20)$$

The complete pN corrections of MBD gravity are given in Eqs.(5)-(7), (15) and (20). Notice that all these solutions are consistent with GR in the limits $V_0 \ll 1$, $|\frac{\varphi^{(2)}}{\phi_0}| \ll 1$ and $\omega_{BD} \to \infty$.

3 Hydrodynamics of Fluid in Massive Brans-Dicke Gravity

Hydrodynamics of fluid in any theory is governed by three laws given by

- Law of conservation of mass,

- Law of conservation of momentum,

- Law of conservation of energy.

In Newtonian theory, these laws are evaluated through equation of continuity and Euler equation of motion. In relativistic hydrodynamics, these law are determined through relativistic (generalized) form of equation of continuity and Euler equation of motion which are obtained from the following identity [3]

$$T^{\alpha\beta}_{;\beta} = 0. \tag{21}$$

3.1 Generalized Equation of Continuity and Euler Equation of motion

According to pN corrections of MBD theory, we have

$$
\begin{aligned}
g_{00} &\approx 1 - \frac{(h_{00}^{(2)})^2}{2} - \frac{1}{2}(\frac{\varphi^{(2)}}{\phi_0})^2 - \frac{2U}{c^2}(\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}}) - 2\Phi - 2\psi, \\
g_{0i} &\approx \frac{1}{c^3}\left(4U_i\left(\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}}\right) - \frac{1}{2}\frac{\partial^2 \chi}{\partial t \partial x_i}\right), \\
g_{ij} &\approx [-1 - \frac{2\gamma_{BD}U}{c^2} - \frac{\Lambda_{BD}r^2}{3c^2}]\delta_{ij}.
\end{aligned}
$$

The resulting Christoffel symbols are given in Appendix **A**. The time component of Eq.(21) gives

$$\frac{\partial T^{00}}{\partial x_0} + \frac{\partial T^{0i}}{\partial x_i} + \left(\Gamma^0_{00} + z_0\right)T^{00} + \left(2\Gamma^0_{0i} + z_i\right)T^{0i} + \Gamma^0_{ij}T^{ij} = 0,$$

which yields

$$
\begin{aligned}
&\frac{\partial \eta}{\partial t} + \frac{\partial \eta v_i}{\partial x_i} - \frac{\rho}{c^2}\left(\frac{3\exp^{-m_0} r}{3 + 2\omega_{BD} + e^{-m_0 r}}\frac{\partial U}{\partial t}\right. \\
&+ v_i\frac{\partial}{\partial x_i}(\frac{3e^{-m_0 r}}{3 + 2\omega_{BD} + e^{-m_0 r}}U) + \frac{\Lambda_{BD}r^2}{3}\right) + \frac{1}{c^2}\left(\rho\frac{\partial U}{\partial t} - \frac{\partial p}{\partial t}\right) \\
&+ v_i\frac{\partial}{\partial x_i}\left(\frac{e^{-m_0 r}}{3 + 2\omega_{BD} + e^{-m_0 r}}U\right) + \frac{\Lambda_{BD}r^2}{3}\right) = 0, \tag{22}
\end{aligned}
$$

where

$$\eta = \rho\left[1 + \frac{1}{c^2}(v^2 + 2U - \frac{2\Lambda_{BD}r^2}{3} + \pi + \frac{p}{\rho})\right].$$

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This equation can replace the equation of continuity of Newtonian gravity in MBD hydrodynamics. The spatial components of Eq.(21) become

$$\frac{1}{c}\frac{\partial T^{0i}}{\partial t} + \frac{\partial T^{ij}}{\partial x_j} + \Gamma^i_{00}T^{00} + 2\Gamma^i_{j0}T^{0j} + z_0 T^{i0} + \Gamma^i_{jk}T^{jk} + z_j T^{ij} = 0,$$

which can be written as

$$\frac{\partial \eta v_i}{\partial t} + \frac{\partial \eta v_i v_j}{\partial x_i} + \frac{\partial}{\partial x_i}\left[\left(1 + 2\gamma_{BD}U + \frac{\Lambda_{BD}r^2}{3}\right)p\right] + \frac{2\rho}{c^2}\frac{d}{dt}\left[(2\gamma_{BD}U\right.$$

$$\left. + \frac{\Lambda_{BD}r^2}{3}\right)v_i\right] - \frac{4\rho}{c^2}\frac{d}{dt}\left[\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}}U_i\right]$$

$$- \frac{\rho}{c^2}\left[\frac{(3 + 2\omega_{BD} + e^{-m_0 r})\sigma}{3 + 2\omega_{BD}}\frac{\partial}{\partial x_i}\left(\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}}U\right) + \frac{\partial \Phi'}{\partial x_i}\right]$$

$$- \frac{4\rho}{c^2}\frac{\partial}{\partial x_j}\left(\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}}U_i\right) - \frac{\rho}{c^2}\frac{\partial}{\partial x_i}\left(\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}}U\right)$$

$$+ \frac{\rho}{2c^2}\left(U_i - U_{\alpha;i\alpha}\right) - \frac{\rho}{2c^2}W_i + \frac{\rho}{2c^2}Z_{i(BD)} = 0, \tag{23}$$

where the potential functions $U_{\alpha;i\alpha}$, W_i and $Z_{i(BD)}$ are described in Appendix **A**. Equations (22) and (23) provide generalized form of equation of continuity and Euler equation of Newtonian hydrodynamics which represent equation of motion in MBD gravity.

3.2   Conservation Laws

Here we discuss hydrodynamics of fluid by evaluating the basic conservation laws in pN limits of MBD gravity [3, 14].

3.2.1   The Conservation of Mass

The conservation of mass implies that mass neither created nor destroyed. The conserved mass can be calculated by integrating Eq.(22) over the volume occupied by the MBD fluid. The resulting equation of continuity is given by

$$\frac{\partial \rho'}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho' v_i\right) = 0, \tag{24}$$

where

$$\rho' = \rho\left[1 + \frac{1}{c^2}\left[\frac{1}{2}v^2 - \frac{\Lambda_{BD}r^2}{3} + \frac{9 + 6\omega_{BD} - e^{-m_0 r}}{3 + 2\omega_{BD} + e^{-m_0 r}}U\right]\right].$$

This equation shows that the mass function expressed in terms of density ρ' remains conserved.

3.2.2 The Conservation of Momentum

The conservation equations of momentum help us to discuss dynamics of fluid motion. The conservation law of linear momentum states that the rate of change of total linear momentum is zero, i.e., the total linear momentum of the system remains constant. For conserved linear momentum, we integrate Eq.(23) over the volume along with boundary condition $p = 0$ so that the conservation equation of total linear momentum is

$$\frac{d}{dt} \int_{v'} L_i dx = 0,$$

or

$$\int_{v'} L_i dx = constant,$$

where L_i is the total linear momentum per unit volume given by

$$
\begin{aligned}
L_i &= \eta v_i + \frac{\rho}{2c^2} \left(U_i - U_{\alpha;i\alpha} \right) + \frac{2\rho}{c^2} \left(\left(2U + \frac{\Lambda_{BD} r^2}{3} \right) v_i \right. \\
&\quad - \left. 2 \left(\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}} U_i \right) \right).
\end{aligned}
\tag{25}
$$

According to law of conservation of angular momentum in homogenous space, the total angular momentum of the system remains constant. This is evaluated by first multiplying the equation of motion by x_j and then subtracted it by the same expression with indices i and j interchanged. The resultant is then integrated over the volume v' giving the equation of conserved total angular momentum as

$$\frac{d}{dt} \int_{v'} J_{ij} dx = 0,$$

or

$$\int_{v'} J_{ij} = constant,$$

with total angular momentum

$$J_{ij} = x_i L_j - x_j L_i. \tag{26}$$

3.2.3   The Conservation of Energy

The law of conservation of energy states that energy neither created nor destroyed, i.e., total energy of the system remains constant. In order to evaluate total energy of the system, we contract Eq.(23) with v_i and then integrate over the volume v'. Consequently, the conservation of energy is

$$\frac{d}{dt} \int_{v'} E dx = 0,$$

or

$$E = constant,$$

where the total energy is given by

$$
\begin{aligned}
E &= \left(\eta - \frac{1}{2}\rho'\right)v^2 + \rho'\Pi - \frac{1}{2}\rho'\left(\frac{3 + 2\omega_{BD}U'}{3 + 2\omega_{BD} + e^{-m_0 r}}\right) + \frac{1}{c^2}\rho\left(-\frac{v^2}{8}\right. \\
&+ \frac{1}{2}\left(\frac{3 + 2\omega_{BD}U}{3 + 2\omega_{BD} + e^{-m_0 r}}\right)^2 - \pi\left(\frac{3 + 2\omega_{BD}U}{3 + 2\omega_{BD} + e^{-m_0 r}}\right) \\
&- \frac{1}{2}v^2\pi - \frac{3}{2}\left(\frac{3 + 2\omega_{BD}v^2 U}{3 + 2\omega_{BD} + e^{-m_0 r}}\right) + 2v^2\left(2\gamma_{BD}U + \frac{\Lambda_{BD}r^2}{3}\right) \\
&+ \frac{1 + 2\omega_{BD} + e^{-m_0 r}}{4(3 + \omega_{BD} + e^{-m_0 r})}v_i U_i - \frac{1}{4}v_i U_{\alpha;i\alpha}\right) + \frac{\rho_{BD}}{c^2}U_{i(BD)}. \qquad (27)
\end{aligned}
$$

Here, we assume super-potential function U' as

$$
\begin{aligned}
\nabla^2\left(\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}}\right)U' &= -4\pi G_{eff}\left(\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}}\right)\rho' \\
&= -4\pi G_{eff}\left(\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}}\right)\rho\left[1 + \frac{1}{c^2}\left[\frac{1}{2}v^2 - \frac{2\Lambda_{BD}r^2}{3}\right.\right. \\
&+ \frac{9 + 6\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}}U\left]\right].
\end{aligned}
$$

4   Stability in Massive Brans-Dicke Gravity

The hydrostatic equilibrium is the state of fluid in which the pressure gradient forces are balanced by all other forces (like gravitational forces) and equation of motion does not depend upon time ($v_i = 0$). Thus, the hydrostatic

conditions from Eq.(23)-(27) are given by

$$\frac{\partial p}{\partial x_i} = \rho g_{BD}, \tag{28}$$

$$\rho' = \rho \left[1 + \frac{1}{c^2} \left[-\frac{\Lambda_{BD} r^2}{3} + \frac{9 + 6\omega_{BD} - e^{-m_0 r}}{3 + 2\omega_{BD} + e^{-m_0 r}} U \right] \right] = Constant, \tag{29}$$

$$L_i = \eta v_i + \frac{\rho}{2c^2} (U_i - U_{\alpha;i\alpha}) + \frac{2\rho}{c^2} \left(\left(2U + \frac{\Lambda_{BD} r^2}{3} \right) v_i \right.$$

$$\left. - 2(\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}} U_i) \right) = 0. \tag{30}$$

$$J_{ij} = 0, \tag{31}$$

$$E = -\frac{1}{2}\rho' \left(\frac{3 + 2\omega_{BD} U'}{3 + 2\omega_{BD} + e^{-m_0 r}} \right) + \frac{1}{2c^2}\rho \left(\frac{3 + 2\omega_{BD} U}{3 + 2\omega_{BD} + e^{-m_0 r}} \right)^2$$

$$- \pi \left(\frac{3 + 2\omega_{BD} U}{3 + 2\omega_{BD} + e^{-m_0 r}} \right). \tag{32}$$

with

$$g_{BD} = \left[\left(1 + 2\gamma_{BD} U + \frac{\Lambda_{BD} r^2}{3} \right) \right]^{-1} \left[\frac{1}{c^2} \left[\frac{(3 + 2\omega_{BD} + e^{-m_0 r})\sigma}{3 + 2\omega_{BD}} \right. \right.$$

$$\times \quad \frac{\partial}{\partial x_i} \left(\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}} U \right) + \frac{\partial \Phi'}{\partial x_i} \right] - \frac{1}{c^2} \left[\frac{\partial}{\partial x_i} \right.$$

$$\times \quad \left(\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}} U \right) - \frac{p}{\rho} \frac{\partial}{\partial x_i} \left[\left(1 + 2\gamma_{BD} U + \frac{\Lambda_{BD} r^2}{3} \right) \right] \right]$$

$$- \quad \frac{1}{2c^2} Z'_{i(BD)} \right],$$

where the value of $Z'_{i(BD)}$ is given in Appendix **A**. The term "ρg_{BD}" represents total gravitational effects of MBD systems in hydrostatic equilibrium. Equations (28)-(32) are stability conditions which describe that in stable configuration (hydrostatic equilibrium) the total MBD gravitational force balances the pressure gradient force due to matter distribution. The total density ρ' due to matter as well as scalar field distribution becomes constant (the fluid apparently becomes incompressible). The total linear as well as angular momentum of the system vanishes.

The above discussion implies that in unstable configuration such as gravitational collapse, the force of gravity takes over the pressure force (hydrostatic equilibrium of the system is disturbed). The total density of the system

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does not remain constant and momentum induces into the system. The conditions of such type of instability in MBD fluid can directly be obtained from Eqs.(28)-(32) as

$$\frac{\partial p}{\partial x_i} \; < \; \rho g_{BD}, \tag{33}$$

$$\rho' \; = \; \rho \left[1 + \frac{1}{c^2} \left[-\frac{\Lambda_{BD} r^2}{3} + \frac{9 + 6\omega_{BD} - e^{-m_0 r}}{3 + 2\omega_{BD} + e^{-m_0 r}} U \right] \right] \neq Constant, \tag{34}$$

$$L_i \; = \; \eta v_i + \frac{\rho}{2c^2} \left(U_i - U_{\alpha;i\alpha} \right) + \frac{2\rho}{c^2} \left(\left(2U + \frac{\Lambda_{BD} r^2}{3} \right) v_i \right.$$

$$- \; 2 \left(\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}} U_i \right) \right) \neq 0.$$

$$J_{ij} \; \neq \; 0, \tag{35}$$

$$E \; \neq \; -\frac{1}{2}\rho' \left(\frac{3 + 2\omega_{BD} U'}{3 + 2\omega_{BD} + e^{-m_0 r}} \right) + \frac{1}{2c^2}\rho \left(\frac{3 + 2\omega_{BD} U}{3 + 2\omega_{BD} + e^{-m_0 r}} \right)^2$$

$$- \; \pi \left(\frac{3 + 2\omega_{BD} U}{3 + 2\omega_{BD} + e^{-m_0 r}} \right). \tag{36}$$

5 Spherically Symmetric Barotropic Stars in Massive Brans-Dicke Gravity

Barotropic is a state of fluid in which total density of a system is a function of pressure only. In homogenous and isotropic universe (pN regimes), barotropic fluid satisfies the equation of state

$$p = w\rho, \tag{37}$$

where w is the equation of state parameter. Different values of w represent different cosmic configurations such as $w = 0$ shows dust, $w = 1/3$ represents radiation era, $w = -1/3$ indicates cosmic string, $w = -2/3$ expresses domain walls and $w < -1/3$ represents dark energy dominated era. It is well-known that the configuration of spherically symmetric stars depends upon time and radial components only. Here, we represent dynamics of a spherical symmetric barotropic stars in pN approximation of MBD gravity.

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5.1 Hydrodynamics

Any spherically symmetric star in MBD gravity has the following hydrodynamical quantities

- the conserved mass function is defined in terms of density

$$\rho' = \rho \left[1 + \frac{1}{c^2} \left[\frac{1}{2}v^2 - \frac{\Lambda_{BD}r^2}{3} + \frac{9 + 6\omega_{BD} - e^{-m_0 r}}{3 + 2\omega_{BD} + e^{-m_0 r}}U \right] \right], \tag{38}$$

- the total conserved linear momentum is given by

$$\begin{aligned}
L_r &= \eta v_r + \frac{\rho}{2c^2}\left(U_r - U_{\alpha;r\alpha}\right) + \frac{2\rho}{c^2}\left(\left(2U + \frac{\Lambda_{BD}r^2}{3}\right)v_r \right.\\
&\quad - \left. 2\left(\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}}U_r\right)\right),
\end{aligned} \tag{39}$$

- the non-zero components of conserved angular momentum are

$$J_{r\theta} = -\theta L_r, \quad J_{r\phi'} = -\phi' L_r, \tag{40}$$

- the total conserved energy of the system is described as

$$\begin{aligned}
E &= \left(\eta - \frac{1}{2}\rho'\right)v^2 + \rho'\Pi - \frac{1}{2}\rho'\left(\frac{3 + 2\omega_{BD}U'}{3 + 2\omega_{BD} + e^{-m_0 r}}\right) + \frac{1}{c^2}\rho\left(-\frac{v^2}{8}\right.\\
&\quad + \frac{1}{2}\left(\frac{3 + 2\omega_{BD}U}{3 + 2\omega_{BD} + e^{-m_0 r}}\right)^2 - \pi\left(\frac{3 + 2\omega_{BD}U}{3 + 2\omega_{BD} + e^{-m_0 r}}\right)\\
&\quad - \frac{1}{2}v^2\pi - \frac{3}{2}\left(\frac{3 + 2\omega_{BD}v^2 U}{3 + 2\omega_{BD} + e^{-m_0 r}}\right) + 2v^2\left(2\gamma_{BD}U + \frac{\Lambda_{BD}r^2}{3}\right)\\
&\quad + \frac{1 + 2\omega_{BD} + e^{-m_0 r}}{4(3 + \omega_{BD} + e^{-m_0 r})}v_r U_r - \frac{1}{4}v_r U_{\alpha;r\alpha}\right) + \frac{\rho_{BD}}{c^2}U_{r(BD)}. \tag{41}
\end{aligned}$$

5.2 Conditions of Stability and Instability

Any star remains stable as long as its hydrostatic equilibrium remains stable. In hydrostatic equilibrium, a spherically symmetric barotropic star satisfies the following conditions

$$\frac{dp}{dr} = \rho g'_{BD} \tag{42}$$

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with

$$g'_{BD} = \left[1 + 2\gamma_{BD}U + \frac{\Lambda_{BD}r^2}{3}\right]^{-1} \left[\frac{1}{c^2}\left[\frac{(3 + 2\omega_{BD} + e^{-m_0r})\sigma}{3 + 2\omega_{BD}}\right.\right.$$

$$\times \frac{d}{dr}\left(\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0r}}U\right) + \frac{d\Phi'}{dr}\right] - \frac{1}{c^2}\frac{d}{dr}\left(\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0r}}U\right)$$

$$- p\frac{d}{dr}\left[1 + 2\gamma_{BD}U + \frac{\Lambda_{BD}r^2}{3}\right] - \frac{1}{2c^2}Z'_{r(BD)}\right].$$

$$\rho' = \rho\left[1 + \frac{1}{c^2}\left[-\frac{\Lambda_{BD}r^2}{3} + \frac{9 + 6\omega_{BD} - e^{-m_0r}}{3 + 2\omega_{BD} + e^{-m_0r}}U\right]\right] = Constant$$

(43)

$$L_r = J_{r\theta} = J_{r\phi'} = 0.$$

(44)

From Eqs.(37), (42) and (43) we get

$$\rho = e^{(\frac{1}{w}\int g''_{BD}dr)},$$

(45)

$$p = we^{(\frac{1}{w}\int g''_{BD}dr)},$$

(46)

$$\rho' = e^{(\frac{1}{w}\int g''_{BD}dr)}\left[1 + \frac{1}{c^2}\left[-\frac{\Lambda_{BD}r^2}{3} + \frac{9 + 6\omega_{BD} - e^{-m_0r}}{3 + 2\omega_{BD} + e^{-m_0r}}U\right]\right] = Constant,$$

(47)

where

$$g''_{BD} = \left[1 + 2\gamma_{BD}U + \frac{\Lambda_{BD}r^2}{3}\right]^{-1} \left[\frac{1}{c^2}\left[\frac{(3 + 2\omega_{BD} + e^{-m_0r})\sigma}{3 + 2\omega_{BD}}\right.\right.$$

$$\times \frac{d}{dr}\left(\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0r}}U\right) + \frac{d\Phi'}{dr}\right] - \frac{1}{c^2}\frac{d}{dr}\left(\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0r}}U\right)$$

$$- w\frac{d}{dr}\left[1 + 2\gamma_{BD}U + \frac{\Lambda_{BD}r^2}{3}\right] - \frac{1}{2c^2}Z''_{r(BD)}\right].$$

The value of $Z''_{r(BD)}$ is given in Appendix **A**.

Since in the phenomenon of gravitational collapse pressure gradient forces are over take by gravitational forces. A barotropic star becomes unstable (collapses) whenever

$$w\frac{d\rho}{dr} < \rho g''_{BD}$$

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which implies

$$\rho \;<\; e^{(\frac{1}{w}\int g''_{BD}dr)}, \tag{48}$$

$$p \;<\; we^{(\frac{1}{w}\int g''_{BD}dr)}, \tag{49}$$

$$\rho' \;<\; e^{(\frac{1}{w}\int g''_{BD}dr)}\left[1+\frac{1}{c^2}\left[-\frac{\Lambda_{BD}r^2}{3}+\frac{9+6\omega_{BD}-e^{-m_0 r}}{3+2\omega_{BD}+e^{-m_0 r}}U\right]\right]. \tag{50}$$

5.3 Cosmic Strings

Cosmic strings are 1-dimensional topological defects that are related to solitonic solutions of the classical equations for complex scalar field. These types of matter satisfy equation of state $p = -1/3\rho$ [16]. They are considered to have immense density and are significant source of gravitational waves. The hydrodynamics of this kind of fluid in MD gravity are described by Eqs.(38)-(41) along with defined equation of state. The conditions for instability configuration (gravitational collapse) are

$$\rho \;<\; e^{(-3\int g''_{BD}dr)},$$

$$p \;<\; \frac{-1}{3}e^{(-3\int g''_{BD}dr)},$$

$$\rho' \;<\; e^{(-3\int g''_{BD}dr)}\left[1+\frac{1}{c^2}\left[-\frac{\Lambda_{BD}r^2}{3}+\frac{9+6\omega_{BD}-e^{-m_0 r}}{3+2\omega_{BD}+e^{-m_0 r}}U\right]\right],$$

$$L_r \;\neq\; J_{r\theta} \neq J_{r\phi'} \neq 0.$$

5.4 Domain Wall

Domain walls are 2-dimensional topological defects in different scalar fields. They are topological solitons which are considered as a resultant of spontaneous broken of discrete symmetry. These are perfect fluid types which obey equation of state $p = -2/3\rho$ [17]. It is believed that collision of two such walls violently emit gravitational waves. The equation of state along with Eqs.(38)-(41) represent hydrodynamics of domain wall in pN regimes under the influence of MBD gravity. The respective conditions for unstable configuration are

$$\rho \;<\; e^{(-\frac{3}{2}\int g''_{BD}dr)},$$

$$p \;<\; \frac{-2}{3}e^{(-\frac{3}{2}\int g''_{BD}dr)},$$

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$$\rho' \; < \; e^{(-\frac{3}{2}\int g''_{BD}dr)}\left[1+\frac{1}{c^2}\left[-\frac{\Lambda_{BD}r^2}{3}+\frac{9+6\omega_{BD}-e^{-m_0 r}}{3+2\omega_{BD}+e^{-m_0 r}}U\right]\right],$$

$$L_r \; \neq \; J_{r\theta} \neq J_{r\phi'} \neq 0.$$

5.5 Dust Fluid

Dust is a state of fluid in which fluid particles are either approximately stationary or moving in non-intersecting geodesics producing zero pressure. Equation (38)-(41) along with $p = 0$ describe hydrodynamics of dust in spherically symmetric configuration. In this case the condition of stability (hydrostatic equilibrium) can be directly obtained from Eq.(42) as follows

$$\rho g''_{BD} = 0,$$

since $\rho \neq 0$, this implies $g''_{BD} = 0$ which yield

$$
\begin{aligned}
Z'''_{r(BD)} \; &= \; \left[\frac{(3+2\omega_{BD}+e^{-m_0 r})\sigma}{3+2\omega_{BD}}\frac{d}{dr}\left(\frac{3+2\omega_{BD}}{3+2\omega_{BD}+e^{-m_0 r}}U\right)+\frac{d\Phi'}{dr}\right] \\
&- \; \frac{d}{dr}\left(\frac{3+2\omega_{BD}}{3+2\omega_{BD}+e^{-m_0 r}}U\right),
\end{aligned}
$$

where the value of $Z'''_{r(BD)}$ is given in Appendix **A**.

6 Conclusions

This paper is devoted to investigate hydrodynamics as well as hydrostatic of MBD fluid in pN regime. For this purpose, we have used simple perturbation scheme and evaluated complete pN approximation of the field equations solution in terms of potential functions of celestial masses. We have approximated g_{00} to $O(c^{-4})$, g_{0i} upto $O(c^{-3})$, g_{ij} to $O(c^{-2})$ and perturbed scalar field upto $O(c^{-2})$. We have generalized standard Newtonian equation of continuity and Euler equation of motion in pN limits of MBD gravity. The equations governing hydrodynamics, stability and instability of the fluid in weak-field regimes are developed. These developed models are then used to discuss stability conditions of a spherically symmetric star in pN approximation of MBD gravity. In particular, we have discussed conditions for unstable configuration of barotropic systems such as dust, cosmic string and domain walls.

The obtained results provide a range of deviation of MBD theory from BD and GR gravities in pN approximation. The governing equations involve some extra and generalized potential functions that are not used by BD and GR theories, which implies that celestial objects used in MBD theory are more massive than those described by GR and BD gravities [3, 14]. It is also found that MBD gravity reduces to GR theory by freezing the dynamics at $\omega_{BD} \to \infty, V_0 \ll 1$ and $| \frac{\varphi^{(2)}}{\phi_0} | \ll 1$. The derived model provides an essential (boot-strap) character of modified gravity (MBD gravity) which can investigate effects of dark energy on the hydrodynamics behavior of large-scale systems or we can explore large-scale systems in accelerated expanding universe.

Appendix A

The contravariant and covariant components of four-velocity and energy-momentum tensor are

$$
\begin{aligned}
u^0 &= 1 + \frac{1}{c^2}(\frac{1}{2}v^2 + 2U - \frac{\Lambda_{BD}r^2}{3}) + O(4), \\
u_0 &= 1 + \frac{1}{c^2}(\frac{1}{2}v^2 - 2U + \frac{\Lambda_{BD}r^2}{3}) + O(4), \\
u^i &= \left[1 + \frac{1}{c^2}(\frac{1}{2}v^2 + 2U - \frac{\Lambda_{BD}r^2}{3})\right]\frac{v_i}{c} + O(4), \\
u_i &= -\frac{v_i}{c} + O(3), \\
T_{00} &= \rho c^2 \left[1 + \frac{1}{c^2}(v^2 - 2U + \frac{2\Lambda_{BD}r^2}{3} + \pi)\right] + O(2), \\
T^{00} &= \rho c^2 \left[1 + \frac{1}{c^2}(v^2) + 2U - \frac{2\Lambda_{BD}r^2}{3} + \pi\right] + O(2), \\
T_{0i} &= -\rho c v_i + O(1), \\
T^{0i} &= \rho c \left[1 + \frac{1}{c^2}(v^2 + 2U - \frac{2\Lambda_{BD}r^2}{3} + \pi + \frac{p}{\rho})\right]v_i + O(3), \\
T_{ij} &= \rho v^i v^j + \delta_{ij}p + O(2), \\
T^{ij} &= \rho v^i v^j + p\delta_{ij} + \frac{1}{c^2}\left[\rho(v^2 + 2U - \frac{2\Lambda_{BD}r^2}{3} + \pi + \frac{p}{\rho})v_i v_j \right.
\end{aligned}
$$

$$- \quad 2p\gamma_{BD}U\delta_{ij} - \frac{2\Lambda_{BD}r^2}{3}\delta_{ij}p\bigg] + O(4).$$

The Christoffel symbols in pN corrections are given by

$$\Gamma^0_{00} = -\frac{1}{c^3}\bigg(\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}}\bigg)\frac{\partial U}{\partial t},$$

$$\Gamma^0_{0i} = -\frac{1}{c^2}\frac{\partial}{\partial x_i}\bigg(\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}}U\bigg),$$

$$\Gamma^0_{ij} = \frac{1}{2c^3}\bigg[2\delta_{ij}\frac{\partial\gamma_{BD}U}{\partial t} + 4\bigg((\frac{\partial}{\partial x_j})(\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}}U_i)$$

$$+ \quad (\frac{\partial}{\partial x_i})(\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}}U_j)\bigg) - \frac{\partial^2\chi}{\partial t\partial x_i\partial x_j}\bigg],$$

$$\Gamma^i_{00} = -\frac{1}{c^2}\frac{\partial}{\partial x_i}\bigg(\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}}U\bigg) + \frac{1}{c^4}\bigg[-\frac{\partial}{\partial x_i}\bigg(\frac{1}{2}(h^{(2)}_{00})^2$$

$$- \quad \frac{1}{2}(\frac{\varphi^{(2)}}{\phi_0})^2 - \Phi - \psi\bigg) - 4\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}}\frac{\partial U_i}{\partial t} + \frac{1}{2}\frac{\partial^2\chi}{\partial t\partial x_i}\bigg],$$

$$\Gamma^i_{0j} = \frac{1}{c^3}\bigg[\gamma_{BD}\frac{\partial U}{\partial t}\delta_{ij} - 2\bigg((\frac{\partial}{\partial x_j})(\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}}U_i)$$

$$+ \quad (\frac{\partial}{\partial x_i})(\frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}}U_j)\bigg)\bigg],$$

$$\Gamma^i_{jk} = \frac{1}{2c^2}\bigg[\frac{\partial}{\partial x_k}(2\gamma_{BD} + \frac{\Lambda_{BD}r^2}{3})\delta_{ij} + \frac{\partial}{\partial x_j}(2\gamma_{BD} + \frac{\Lambda_{BD}r^2}{3})\delta_{ik}$$

$$- \quad \frac{\partial}{\partial x_i}(2\gamma_{BD} + \frac{\Lambda_{BD}r^2}{3})\delta_{jk}\bigg],$$

and

$$\Gamma^\mu_{\alpha\mu} = z_\alpha = \frac{\partial(log\sqrt{-g})}{\partial x_\alpha} = \frac{2}{c^2}\frac{\partial}{\partial x_\alpha}\bigg[2U\gamma_{BD} + \frac{\Lambda_{BD}r^2}{3}\bigg],$$

which gives

$$\Gamma^\mu_{0\mu} = z_0 = \frac{2}{c^3}\frac{\partial(\gamma_{BD}U)}{\partial t}, \quad \Gamma^\mu_{i\mu} = z_i = \frac{\partial}{\partial x_i}\bigg[2\gamma_{BD}U + \frac{\Lambda_{BD}r^2}{3}\bigg]. \qquad (51)$$

The potential functions $U_{\alpha;i\alpha}$, $W_i(\mathbf{x})$ and $Z_{i(BD)}$ expressed in generalized Euler equation of motion are defined by

$$U_{\alpha;i\alpha} = G_{eff}\int_v \rho(\mathbf{x}')v_\alpha(\mathbf{x}')\frac{(x_i - x_i')(x_i - x_i')dx'}{|\mathbf{x} - \mathbf{x'}|^3},$$

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$$
\begin{aligned}
W_i(\mathbf{x}) &= v_\alpha \frac{\partial}{\partial x_\alpha}(U_i - U_{j;ij}) = -G_{eff} \int_v \rho(\mathbf{x}') v_i(\mathbf{x}) v_i(\mathbf{x}') \frac{(x_i - x_i')dx'}{|\mathbf{x} - \mathbf{x}'|^3} \\
&- G_{eff} \int_v \rho(x') \left[v_i(\mathbf{x}) v_\alpha(\mathbf{x}') + v_i(\mathbf{x}') v_\alpha(\mathbf{x}) \right] \frac{(x_\alpha - x_\alpha')dx'}{|\mathbf{x} - \mathbf{x}'|^3} \\
&+ 3G_{eff} \int_v \rho(\mathbf{x}') \left[v_\alpha(\mathbf{x}) v_\beta(\mathbf{x}')(x_\alpha - x_\alpha')(x_\beta - x_\beta') \right] \frac{x_i - x_i'}{|\mathbf{x} - \mathbf{x}'|^5},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\rho}{2c^2} Z_{i(BD)} &= \frac{\rho_{BD}}{c^2} \frac{\partial U_{BD}}{\partial x_i} = \rho \left[-2 \left(1 + \frac{e^{-m_0 r}}{3 + 2\omega_{BD} + e^{-m_0 r}} U \right) \right. \\
&+ \frac{\Lambda_{BD} r^2}{3} + \frac{2\rho v^2}{c^2} \left(\frac{\partial U}{\partial x_i} + \frac{\partial}{\partial x_i} \left(\frac{e^{-m_0 r}}{3 + 2\omega_{BD} + e^{-m_0 r}} U \right) \right. \\
&+ \frac{\partial (\Lambda_{BD} U)}{\partial x_i} + \frac{\partial \Lambda_{BD} r^2}{\partial x_i} \right) + \frac{p}{\rho c^2} \left(2\frac{\partial U}{\partial x_i} \right. \\
&+ 2\frac{\partial}{\partial x_i} \left(\frac{e^{-m_0 r}}{3 + 2\omega_{BD}} U \right) + 2\frac{\partial (\Lambda_{BD} U)}{\partial x_i} + \frac{1}{2}\frac{\partial \Lambda_{BD} r^2}{\partial x_i} \right) \\
&+ \left. \left(U - \frac{\Lambda_{BD} r^2}{6} \right) \frac{\partial \Lambda_{BD} r^2}{\partial x_i} \right].
\end{aligned}
$$

The values of $Z'_{i(BD)}$, $Z''_{r(BD)}$ and $Z'''_{r(BD)}$ are given by

$$
\begin{aligned}
\frac{\rho}{2c^2} Z'_{i(BD)} &= \frac{\rho'_{BD}}{c^2} \frac{\partial U_{BD}}{\partial x_i} = \rho \left[-\frac{2}{c^2} \left(1 + \frac{e^{-m_0 r}}{3 + 2\omega_{BD} + e^{-m_0 r}} U \right) + \frac{\Lambda_{BD} r^2}{3c^2} \right. \\
&+ \frac{p}{\rho c^2} \left(2\frac{\partial U}{\partial x_i} + 2\frac{\partial}{\partial x_i} \left(\frac{e^{-m_0 r}}{3 + 2\omega_{BD} + e^{-m_0 r}} U \right) + 2\frac{\partial (\gamma_{BD} U)}{\partial x_i} \right. \\
&+ \frac{1}{2}\frac{\partial \Lambda_{BD} r^2}{\partial x_i} \right) + \left. \left(U - \frac{\Lambda_{BD} r^2}{6} \right) \frac{\partial \Lambda_{BD} r^2}{\partial x_i} \right], \\
\frac{\rho}{2c^2} Z''_{r(BD)} &= \frac{\rho'_{BD}}{c^2} \frac{\partial U_{BD}}{\partial x_i} = \rho \left[-\frac{2}{c^2} \left(1 + \frac{e^{-m_0 r}}{3 + 2\omega_{BD} + e^{-m_0 r}} U \right) + \frac{\Lambda_{BD} r^2}{3c^2} \right. \\
&+ \frac{w}{c^2} \left(2\frac{dU}{dr} + 2\frac{d}{dr} \left(\frac{e^{-m_0 r}}{3 + 2\omega_{BD} + e^{-m_0 r}} U \right) + 2\frac{d(\gamma_{BD} U)}{dr} \right. \\
&+ \frac{1}{2}\frac{d\Lambda_{BD} r^2}{dr} \right) + \left. \left(U - \frac{\Lambda_{BD} r^2}{6} \right) \frac{d\Lambda_{BD} r^2}{dr} \right], \\
Z'''_{r(BD)} &= \left[-\frac{2}{c^2} \left(1 + \frac{e^{-m_0 r}}{3 + 2\omega_{BD} + e^{-m_0 r}} U \right) + \frac{\Lambda_{BD} r^2}{3c^2} \right. \\
&+ \left. \left(U - \frac{\Lambda_{BD} r^2}{6} \right) \frac{d\Lambda_{BD} r^2}{dr} \right].
\end{aligned}
$$

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